Kinetic Equations

Solution to the Exercises

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Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$ be bounded with $\nabla_z b \in L^\infty\left([0,T]\times\mathbb{R}^d; M_d(\mathbb{R})\right)$.

Recall that given a measure μ on $\mathbb{R}^d$ and a measurable function φ we use the following notation:

$$\mu(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x). \tag{1}$$

Consider the *Liouville equation* given by

$$\begin{cases} \partial_t \mu_t(\psi) = \mu_t\left(b(t,\cdot)\cdot\nabla_z\psi\right), & \forall t\in[0,T],\ \forall\psi\in C_c^\infty\left(\mathbb{R}^d\right), \\ \mu_t(\psi)|_{t=0} = \mu_0(\psi), & \forall\psi\in C_c^\infty\left(\mathbb{R}^d\right). \end{cases} \tag{2}$$

Suppose that $M = \{\mu_t \mid t\in[0,T]\}$ is a family of measures such that for each $t\in[0,T]$ the measure μ_t is $\mathcal{L}^d$ absolutely continuous (we will write equivalently $\mathcal{L}^d$-a.c., where $\mathcal{L}^d$ is the Lebesgue measure in d dimensions); moreover, assume that exists $f_M \in C^1\left([0,T]\times\mathbb{R}^d\right)$ such that $d\mu_t(z) = f_M(t,z)\,dz$.

Under these assumptions, prove that M is a solution to (2) if and only if f_M is a classical solution to

$$\begin{cases} \partial_t f + \mathrm{div}_z(bf) = 0, \\ f(t,z)|_{t=0} = f_M(0,z), \quad \forall z\in\mathbb{R}^d. \end{cases} \tag{3}$$

Remark. *Notice that if $d = 6$, $z = (x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ and $b(t,x,v) = (v, E(t,x))$ the Liouville equation is written as*

$$\begin{cases} \partial_t \mu_t(\psi) = \mu_t\left(v\cdot\nabla_x\psi + E(t,x)\cdot\nabla_v\psi\right), & \forall t\in[0,T],\ \forall\psi\in C_c^\infty\left(\mathbb{R}^d\right), \\ \mu_t(\psi)|_{t=0} = \mu_0(\psi), & \forall\psi\in C_c^\infty\left(\mathbb{R}^d\right). \end{cases} \tag{4}$$

Proof. First suppose that M is a solution and fix $\psi\in C_c^\infty\left(\mathbb{R}^d\right)$. As a consequence we have

$$\int_{\mathbb{R}^d}\psi(z)\,\partial_t f_M(t,z)\,dz = \partial_t\left(\int_{\mathbb{R}^d}\psi(z)\,f_M(t,z)\,dz\right) = \int_{\mathbb{R}^d} b(t,z)\cdot\nabla_z\psi(z)\,f_M(t,z)\,dz. \tag{5}$$

Given that f_M is C^1 and that ψ is compactly supported we can integrate by part to get

$$\int_{\mathbb{R}^d}\psi(z)\,\partial_t f_M(t,z)\,dz = \int_{\mathbb{R}^d} b(t,z)\cdot\nabla_z\psi(z)\,f_M(t,z)\,dz \tag{6}$$

$$= \int_{\mathbb{R}^d}\nabla_z\psi(z)\cdot b(t,z)\,f_M(t,z)\,dz \tag{7}$$

$$= -\int_{\mathbb{R}^d}\psi(z)\,\mathrm{div}_z\left(b(t,z)\,f_M(t,z)\right)dz. \tag{8}$$

From the fact that ψ is arbitrary, we get that almost everywhere in z we get

$$\partial_t f_M(t,z) + \operatorname{div}_z\left(b(t,z)\, f_M(t,z)\right) = 0. \tag{9}$$

From the fact that the function f_M is C^1 the equality is moreover pointwise and we get that f_M is a classical solution of (3).

On the other hand, if f_M is a classical solution, we can do the previous steps backwards to get that the measure $d\mu_t(x) := f_M(t,z)\, dz$ is a solution to the Liouville equation (2).

□

Exercise 2

Recall that if $p \in \mathbb{N}$, we defined in class

$$\mathcal{P}_p\left(\mathbb{R}^d\right) := \left\{\mu \text{ Borel measure} |\ \mu \geqslant 0,\ \mu\left(\mathbb{R}^d\right) = 1, \int_{\mathbb{R}^d} |x|^p\, d\mu(x) < +\infty\right\}. \tag{10}$$

Consider now a sequence of measures $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}_1\left(\mathbb{R}^d\right)$ and $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$. We say that μ_k *converges to* μ *weakly* and we write $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ if

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x), \qquad \forall \varphi \in C_{\mathrm{b}}\left(\mathbb{R}^d\right), \tag{11}$$

where $C_{\mathrm{b}}\left(\mathbb{R}^d\right)$ denotes the space of bounded continuous functions.

Prove that the following properties are equivalent:

(i) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) = \int_{\mathbb{R}^d} |x|\, d\mu(x); \tag{12}$$

(ii) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\limsup_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) \leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x); \tag{13}$$

(iii) $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ and

$$\lim_{R\to+\infty} \limsup_{k\to+\infty} \int_{|x|\geqslant R} |x|\, d\mu_k(x) = 0; \tag{14}$$

(iv) For any $\varphi \in C\left(\mathbb{R}^d\right)$ such that there exists a positive constant C with $|\varphi(x)| \leqslant C(1+|x|)$ for all $x \in \mathbb{R}^d$ we have

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x). \tag{15}$$

Proof. We will show the chain of implications $(iv) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ which will prove the exercise.

To prove that $(iv) \Rightarrow (i)$, we get that given that $C_{\mathrm{b}}\left(\mathbb{R}^d\right) \subseteq C\left(\mathrm{R}^d\right)$, (15) implies (11) and therefore we have immediately that $\mu_k \rightharpoonup \mu$ as $k \to +\infty$. Furthermore, $|x| \in C\left(\mathrm{R}^d\right)$ and trivially satisfies the condition in (iv), and therefore usging $\varphi(x) = |x|$ in (15) we get (12).

To prove that $(i) \Rightarrow (ii)$, is enough to notice that if the limit of a sequence exists, then the limsup coincides with it and the equality in (12) implies the inequality in (13).

To prove that $(ii) \Rightarrow (iii)$, fix $R > 0$ and denote with B_r the open ball of radius r centered in 0 in $\mathbb{R}^d$. Using (13) we then have

$$\limsup_{k\to+\infty} \int_{|x|\geqslant R} |x|\, d\mu_k(x) = \limsup_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) - \limsup_{k\to+\infty} \int_{B_R} |x|\, d\mu_k(x) \tag{16}$$

$$\leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x) - \limsup_{k\to+\infty} \int_{B_R} |x|\, d\mu_k(x). \tag{17}$$

We want to use now the weak convergence to control the last term. Given that the caracteristic function of the ball is not continuous, consider a family of functions $f_\varepsilon \in C_{\mathrm{b}}\left(\mathrm{R}^d\right)$, with $\varepsilon > 0$ a positive real number, such that

$$\chi_{B_{R-\varepsilon}}(x) \leqslant f_\varepsilon(x) \leqslant \chi_{B_R}(x), \tag{18}$$

where χ_E represent the indicator function of the measurable set E. Now, $|x| f_\varepsilon(x) \in C_{\mathrm{b}}\left(\mathbb{R}^d\right)$, so from the fact that $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ we get

$$-\limsup_{k\to+\infty} \int_{B_R} |x|\, d\mu_k(x) = -\limsup_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, \chi_{B_R}(x)\, d\mu_k(x) \tag{19}$$

$$\leqslant -\limsup_{k\to+\infty} \int_{\mathbb{R}^d} |x|\, f_\varepsilon(x)\, d\mu_k(x) \tag{20}$$

$$= -\int_{\mathbb{R}^d} |x|\, f_\varepsilon(x)\, d\mu(x) \tag{21}$$

$$\leqslant -\int_{\mathbb{R}^d} |x|\, \chi_{B_{R-\varepsilon}}(x)\, d\mu(x) = -\int_{B_{R-\varepsilon}} |x|\, d\mu(x). \tag{22}$$

Given that $|x| \in L^1\left(\mathbb{R}^d; d\mu\right)$, we get that for any fixed $\varepsilon > 0$ we have

$$\lim_{R\to+\infty} -\int_{B_{R-\varepsilon}} |x|\, d\mu(x) = -\int_{\mathbb{R}^d} |x|\, d\mu(x) \tag{23}$$

and therefore

$$\limsup_{R\to+\infty} \limsup_{k\to+\infty} \int_{|x|\geqslant R} |x|\, d\mu_k(x) \leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x) + \limsup_{R\to+\infty} \left(-\int_{B_{R-\varepsilon}} |x|\, d\mu(x)\right) \tag{24}$$

$$= \int_{\mathbb{R}^d} |x|\, d\mu(x) + \lim_{R\to+\infty} \left(-\int_{B_{R-\varepsilon}} |x|\, d\mu(x)\right) = 0. \tag{25}$$

Therefore the limit on the left exists and is 0, implying (14)

To prove that $(iii) \Rightarrow (iv)$, we consider $\varphi \in C\left(\mathbb{R}^d\right)$ such that there exists a constant C with $|\varphi(x)| \leqslant C(1+|x|)$. Assume first that φ is nonnegative. Fix now $\varepsilon > 0$; we get that

$$\int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) \leqslant \tag{26}$$

$$\leqslant \int_{B_{R-\varepsilon}} \varphi(x)\, d\mu_k(x) - \int_{B_R} \varphi(x)\, d\mu(x) \tag{27}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R} (1+|x|)\, d\mu(x) \tag{28}$$

$$\leqslant \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu_k(x) - \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu(x) \tag{29}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x) \tag{30}$$

$$\leqslant \left|\int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu_k(x) - \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu(x)\right| \tag{31}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x). \tag{32}$$

Proceeding analogously we can also get

$$\int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x) \leqslant \tag{33}$$

$$\leqslant \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu(x) - \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu_k(x) \tag{34}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x) \tag{35}$$

$$\leqslant \left|\int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu_k(x) - \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu(x)\right| \tag{36}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x). \tag{37}$$

Combining the previous estimates we get

$$\left|\int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x)\right| \leqslant \tag{38}$$

$$\leqslant \left|\int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu_k(x) - \int_{\mathbb{R}^d} \varphi(x) f_\varepsilon(x)\, d\mu(x)\right| \tag{39}$$

$$+ C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x). \tag{40}$$

Now $\varphi f_\varepsilon \in C_{\mathrm{b}}\left(\mathbb{R}^d\right)$, so we can use the fact that $\mu_k \rightharpoonup \mu$ as $k \to +\infty$ to get

$$\limsup_{k\to+\infty} \left|\int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu_k(x)\right| \leqslant \tag{41}$$

$$\leqslant C \limsup_{k\to+\infty} \int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu_k(x) + C\int_{|x|\geqslant R-\varepsilon} (1+|x|)\, d\mu(x). \tag{42}$$

The left hand side is independent on R, so we can perform the limit $R \to +\infty$. Given that $|x| \in L^1\left(\mathbb{R}^d; d\mu\right)$ and using (14) we get

$$\limsup_{k\to+\infty}\left|\int_{\mathbb{R}^d}\varphi(x)\,d\mu(x)-\int_{\mathbb{R}^d}\varphi(x)\,d\mu_k(x)\right| = \tag{43}$$

$$=\limsup_{R\to+\infty}\limsup_{k\to+\infty}\left|\int_{\mathbb{R}^d}\varphi(x)\,d\mu(x)-\int_{\mathbb{R}^d}\varphi(x)\,d\mu_k(x)\right| \tag{44}$$

$$\leqslant C\limsup_{R\to+\infty}\limsup_{k\to+\infty}\int_{|x|\geqslant R-\varepsilon}(1+|x|)\,d\mu_k(x) \tag{45}$$

$$+\,C\limsup_{R\to+\infty}\int_{|x|\geqslant R-\varepsilon}(1+|x|)\,d\mu(x)=0. \tag{46}$$

So for any $\varphi \in C\left(\mathbb{R}^d\right)$ such that there exists a constant C with $|\varphi(x)| \leqslant C(1+|x|)$ and positive (15) is proven; for any generic $\varphi \in C\left(\mathbb{R}^d\right)$ such that there exists a constant C with $|\varphi(x)| \leqslant C(1+|x|)$ now is enough to notice that $\varphi_+ = \max\{\varphi, 0\}$ and $\varphi_- = \varphi_+ - \varphi$ we get

$$\varphi = \varphi_+ - \varphi_- \tag{47}$$

$$\varphi_+,\ \varphi_- \in C\left(\mathbb{R}^d\right) \tag{48}$$

$$|\varphi_\pm| \leqslant |\varphi| \leqslant C(1+|x|), \tag{49}$$

so by linearity of integration and limit we get that (15) is true for a generic φ; therefore (iv) is true.

□

Exercise 3

Recall that the space $\mathrm{Lip}\left(\mathbb{R}^d\right)$ is defined as the set of function φ such that $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} < +\infty$, where

$$\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} := \sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{d_1(x,y)}, \tag{50}$$

and where $d_1(\cdot,\cdot)$ is the euclidean distance between two points.

Prove that the function $\mathcal{W}_1 : \mathcal{P}_1\left(\mathbb{R}^d\right) \times \mathcal{P}_1\left(\mathbb{R}^d\right) \to \mathbb{R}$ defined for all $\mu,\ \nu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$ as

$$\mathcal{W}_1(\mu,\nu) := \sup\left\{\mu(\varphi)-\nu(\varphi) \mid \varphi\in \mathrm{Lip}\left(\mathbb{R}^d\right),\ \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)}\leqslant 1\right\}. \tag{51}$$

defines a distance on $\mathcal{P}_1\left(\mathbb{R}^d\right)$.

Proof. Recall that a *distance* on a set function X is a map $d : X \times X \to [0, +\infty)$ such that

(i) $d(x,y) = 0 \Leftrightarrow x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,\ y \in X$;

(iii) $d(x,y) \leqslant d(x,z) + d(z,y)$ for all $x,\ y,\ z \in X$.

We prove these properties one by one. We first notice that for any $\varphi \in \mathrm{Lip}\left(\mathbb{R}^d\right)$ with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$ we get

$$
\begin{aligned}
|\varphi(x)| &\leqslant |\varphi(0)| + |\varphi(x) - \varphi(0)| \leqslant |\varphi(0)| + \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} d_1(x,0) && (52)\\
&\leqslant |\varphi(0)| + \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} |x| \leqslant |\varphi(0)| + |x|. && (53)
\end{aligned}
$$

Therefore for any $\mu,\ \nu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$ we get

$$
\begin{aligned}
\mu(\varphi) - \nu(\varphi) &= \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\nu(x) && (54)\\
&\leqslant 2|\varphi(0)| + \int_{\mathbb{R}^d} |x|\, d\mu(x) + \int_{\mathbb{R}^d} |x|\, d\nu(x) < +\infty && (55)
\end{aligned}
$$

therefore we get that on the one hand $\mathcal{W}_1(\mu,\nu) < +\infty$, on the other given that the function constantly 0 is Lipschitz and $\|0\|_{\mathrm{Lip}(\mathbb{R}^d)} = 0 \leqslant 1$ then $\mathcal{W}_1(\mu,\nu) \geqslant 0$, therefore $\mathcal{W}_1 : \mathcal{P}_1\left(\mathbb{R}^d\right) \times \mathcal{P}_1\left(\mathbb{R}^d\right) \to [0,+\infty)$. We now prove the properties of a distance.

(i) If $\mu = \nu$ then $\mathcal{W}_1(\mu,\nu) = 0$ trivially. Suppose now that $\mu,\ \nu \in P_1\left(\mathbb{R}^d\right)$ such that $\mathcal{W}_1(\mu,\nu) = 0$. Then let $\varphi \in \mathrm{Lip}\left(\mathbb{R}^d\right)$ with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$. Clearly also $-\varphi \in \mathrm{Lip}\left(\mathbb{R}^d\right)$; then we get

$$
|\mu(\varphi) - \nu(\varphi)| = \pm(\mu(\varphi) - \nu(\varphi)) = \mu(\pm\varphi) - \nu(\pm\varphi) \leqslant \mathcal{W}_1(\mu,\nu) = 0, \qquad (56)
$$

therefore $\mu(\varphi) = \nu(\varphi)$ for all $\varphi \in \mathrm{Lip}\left(\mathbb{R}^d\right)$ (in principle with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$, but by linearity of the integral this additional condition can be dropped).

Consider now a function $\psi \in C^1_{\mathrm{c}}\left(\mathbb{R}^d\right)$ such that $\|\psi\|_{L^1(\mathbb{R}^d)} = 1$ and define $\psi_M(x) := M^d\psi(Mx)$; for any measurable set E, define $\chi_{E,M} := \chi_E * \psi_M$. On the one hand, from the regularity of ψ_M we get that $\chi_{E,M}$ is Lipschitz and that $\|\chi_{E,M}\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\psi_M\|_{L^1(\mathbb{R}^d)} = \|\psi\|_{L^1(\mathbb{R}^d)} = 1$. Moreover, $\chi_{E,M} \to \chi_E$ for $M \to +\infty$ almost everywhere. Therefore using (56) and dominated convergence theorem we get that

$$
\begin{aligned}
\mu(E) &= \int_E d\mu(x) = \int_{\mathbb{R}^d} \chi_E(x)\, d\mu(x) = \lim_{M\to+\infty} \int_{\mathbb{R}^d} \chi_{E,M}(x)\, d\mu(x) && (57)\\
&= \lim_{M\to+\infty} \mu(\chi_{E,M}) = \lim_{M\to+\infty} \nu(\chi_{E,M}) = \lim_{M\to+\infty} \int_{\mathbb{R}^d} \chi_{E,M}(x)\, d\nu(x) && (58)\\
&= \int_{\mathbb{R}^d} \chi_E(x)\, d\nu(x) = \int_E d\nu(x) = \nu(E), && (59)
\end{aligned}
$$

and therefore $\mu = \nu$.

(ii) The second property comes from the fact that if φ is Lipschitz with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$, also $-\varphi$ is, therefore the two family on which we consider the supremum to get $\mathcal{W}_1(\mu,\nu)$ and $\mathcal{W}_1(\nu,\mu)$ are identical and therefore $\mathcal{W}_1(\mu,\nu) = \mathcal{W}_1(\nu,\mu)$.

(iii) Consider $\mu,\ \nu,\ \lambda \in \mathcal{P}_1\left(\mathbb{R}^d\right)$. We get that for any φ Lipschitz with $\|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$ we have

$$\mu(\varphi) - \nu(\varphi) = \mu(\varphi) - \lambda(\varphi)\,\lambda(\varphi) - \nu(\varphi) \leqslant \mathcal{W}_1(\mu, \lambda) + \mathcal{W}_1(\lambda, \nu) \tag{60}$$

where in the last step we used the definitions of $\mathcal{W}_1(\mu, \lambda)$ and $\mathcal{W}_1(\lambda, \nu)$. Taking the supremum over φ we get the result.

□